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STATISTICAL CONSIDERATIONS IN CONTROL MODELS WITH APPLICATIONS

by

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Technical Report No. 211
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Many applications of control theory, especially in areas other than engineering, require the estimation of the system equations from available data. Applications from patient care system, effect of air pollution on health and dynamic economic models are discussed. A framework is studied in which the control model can be applied using the data provided by the system itself. Special objective functions are utilized to illustrate the technique.

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0. Introduction

Regulation and control play an important role in many systems. The system performance is generally utilized to direct the system behavior towards the anticipated goal. Such feedback mechanisms are necessary elements of many processes. The immune response of a human body, the air temperature in a house or the administration of drugs to a patient by a physician, exhibit the same feedback process. The theory of such controlled processes is highly developed and has been applied to several areas in scientific research, in engineering, in business systems, and in government operations.

In this paper the elements of a control process are discussed from the point of view of statistical applications. Several important applications are pointed out where the introduction of the feedback mechanism and the development of an optimal control policy is likely to improve the ultimate performance of the process. Examples from patient care in the recovery room, monitoring of air pollutants and dynamic economic models are given.

Usually, models of control theory used in applications assume that the dynamics of the process are completely known and certain function, measuring system performance, is to be optimized. Both deterministic and stochastic models are used in practice and the derivation of optimal control policies often require well known technique of dynamic programming. However, the case when models involve unknown parameters in the model, has not been treated well in the literature.

We provide the framework in which the parameters of the dynamic model can be estimated based on the data generated by the process. Using certain well known algorithms, these estimates are updated as the process develops and the optimal policy is then obtained. It is clear that the policy will heavily depend on the random behavior of the process. These complications cannot be completely avoided. Suggestions are made to use the statistical properties of the estimates which are involved in the dynamic programming solution of the control problem. Linear control process with quadratic cost criterion is used for the purpose of illustration. Such solutions have direct application to the patient care problems which have received wide attention recently.

1. Applications

In this section we consider a few applications where the control theory could be utilized with advantage. The problem of patient care in the recovery room, and the problem of monitoring of air pollutants are discussed. There are many other areas such as in the study of dynamic economic models where the control mechanism is evident. A recent comprehensive account has been given by Chow (1975) for the dynamic economic models.

Patient Monitoring

In monitoring patients in surgery or recovery room, elements of a feedback control process are in evidence. A common procedure in monitoring the well being of a fetus, for example, is to monitor the levels of creatinine in amniotic fluid and extriol excretion in maternal urine. Similarly in the management of pharmacologic intervention or in general patient care, the nurse-patient-physician system acts as a feedback control process, for example see Siefen et al (1979).

The model of patient care as a control process and the resulting dynamic programming solution was discussed by Rustagi (1968). Recent applications of the on-line computers in the administration of patient care has been made in several areas, for reference, see Waxman and Stacy (1965), Hammond, Kirkendall and Calfee, (1979), Sheppard, Kirklin and Kouchoukos (1974), Sheppard, Kouchoukos, Shotts et al (1975), Sheppard and Kouchoukos (1976) and Pryor et al (1975).

We discuss one of these studies in more detail below.

Sheppard and Kouchokos (1976) have provided several situations where the feedback mechanism is practiced by the help of electronic computers. For example the regulation of arterial blood pressure is carried out automatically through monitoring blood pressures using certain closed loop mechanisms. Sheppard, Kirklin and Kouchoukos (1974) have recently demonstrated by actually performing decision making tasks by computers in acutely ill patients. Such decisions have been programmed on a computer so as to monitor patients at the Alabama Medical Center for analysis and treatment of impaired cardiac performance. Table I provides the monitoring logic which is implemented automatically. It should be noticed that the authors have demonstrated the fact that a given set of logical decisions can be performed automatically. These decisions are given in advance and no attempt has been made to obtain the best possible decision in a given situation.

Consider the case where several alternative procedures are known to be practiced by the clinician. It would then be worth while to choose the best possible decision under

Table I.

Logic for Analysis and Treatment of
Impaired Cardiac Performance
Early After Operation

(Sheppard, Kirklin and Kouchokos, 1974)

Mean Left Arterial Pressure (mm Hg)	Mean Arterial Pressure (mm Hg)	Cardiac Index ($\ell/min/m^2$)		
		Less than 2	2-3	More than 3
7 or less	-	Blood	Blood	Blood
7-14	-	Blood	Blood	-
15-18	Less than 100	Epinephrine	-	-
	More than 100	Epinephrine, Dopanine or isoproterenol	Trimethephan or nitroprusside	-
More than 18	Less than 100	same as above	-	-
	More than 100	Epinephrine, Dopamine, or isoproterenol plus trimathaphan or nitroprusside	Trimethaphan or Nitroprusside	-

the assumption of a certain optimality criterion.

It whould be noticed that the feedback process in patient care involves measurements of physiological and other variables which behave in general according to some random phenomena. Hence stochastic control theory models are more appropriate to study the patient care process.

Air Pollution and Environmental Health

Another problem where optimal control theory can be applied usefully, occurs in environmental health. It has been demonstrated that high levels of air pollutants are injurious to health and general wellbeing of living systems. Hence various forms of governmental controls have been established to regulate pollutants in the environment. Legal and punitive action are taken against those who are regarded as responsible for creating this hazardous environment. Large industrial corporations are subjected to such control and regulation by the U. S. Environmental Protection Agency.

In the monitoring for the purposes of regulation of air pollutants, elements of a stochastic control process are evident. According to various Clean Air Acts of the U. S. Government, the standards for the various pollutants, such as carbon dioxide, sulphur dioxide, particulate matter are specified by law. As soon as they exceed certain limits, steps are taken to control the various sources of emission of the pollutants. The structure of a feedback process in the regulation of air pollution, can be easily seen from this process of law enforcement.

National Economic Models

In the study of national economics, certain controls are made by the Federal Reserve Board through the manipulation of prime interest rate as well as through other steps which may affect the money supply. The process of control of the national economy requires the knowledge of the state of the economy in terms of several important variables so as to allow taking appropriate action.

Consider that the economy is described by the total amount of consumer expenditure and private investment expenditure. The control can be exercised by Government expenditure and the total money supply. The optimality criterion in this case can be considered to be minimization of the discrepancy between the growth rates of consumption and private investment expenditure by certain targeted increase of these expenditures. This discrepancy may be formalized by a quadratic criterion. The model utilizing some assumed numbers has been described by Chow (1975). Similar discussion of macroeconomic models with random parameters has been made by Havenner and Graine (1973). For multiple time series, a recent study of the same type is by Bovas (1980).

In the next section we describe the basic models of control theory and discuss the linear model in some detail.

2. Control Model

In most of the engineering and other applications of control theory, a general assumption made is that the dynamics of the system are known. They are generally described by differential or difference equations. Both deterministic and stochastic models are used and lead to interesting problems depending on the type of objective functions used for optimization of controls. There is an extensive literature on control theory, some of which is mentioned in the references here, for example, see Polak (1971), Pshenichnyi (1971), Bertsekes (1976), Gihman and Skorohod (1979).

There are many situations in applications such as in patient care, where the nature of the performance of the system is not realistically described by deterministic models and hence stochastic models must be used to describe such systems. In economics, for example, one is confronted with the problem of obtaining an optimal control policy, when the economic system is being affected by a large number of uncertain factors. Chow (1973) has considered the problem of finding an optimal policy in case of economic dynamic systems. Not only the measurements in such systems are random but also the form of the system performance has to be approximated by some hypothetical model. In econometrics, generally one uses a linear model in associating the output of the system with the input and the control utilized. Using quadratic cost criterion, optimal policies are derived. Besides being computationally feasible, such linear models describe the phenomenon fairly well and have been fully treated in various contexts in the literature. Box and Jenkins (1968) have discussed statistical models for control of time series. We first discuss the case of a deterministic control model. The system is generally described by a state vector $\underset{\sim}{x}_t$ of dimension n at time t. Let $\underset{\sim}{u}_t$ be the control vector in p-dimensions. In the case of discrete time points, t = 0, 1, 2, ..., T, the deterministic control process can be described by the difference equation given below:

$$\underset{\sim}{x}_{t+1} = g_t(\underset{\sim}{x}_t, \underset{\sim}{u}_t), \quad t = 0, 1, 2, \ldots, T-1 \qquad (2.1)$$

where g_t are known functions given for the system. If there is feedback present in the system, the present state of the system is used to guide the system back to its normal operation. That is, we assume that

$$U_{t+1} = h(x_t, z_t) \qquad (2.2)$$

That is, the discrepency of the state vector $\underset{\sim}{x}_t$ from its desired or target value $\underset{\sim}{z}_t$, is used to design the control. The control is chosen in such a way so as to optimize the overall performance, for known k,

$$J(\underset{\sim}{x}_0) = \sum_{t=0}^{T} k(\underset{\sim}{x}_t, \underset{\sim}{z}_t). \tag{2.3}$$

$J(\underset{\sim}{x}_0)$ is implicitly a function of $\underset{\sim}{u}_t$.

Following forms of the function k, are commonly utilized in practice.

(1) $$k(\underset{\sim}{x}_t, \underset{\sim}{z}_t) = (\underset{\sim}{x}_t-\underset{\sim}{z}_t)'(\underset{\sim}{x}_t-\underset{\sim}{z}_t) + a\, \underset{\sim}{u}'_t\, \underset{\sim}{u}_t, \tag{2.4}$$

(2) $$k(\underset{\sim}{x}_t, \underset{\sim}{z}_t) = \begin{cases} a\, \underset{\sim}{u}'_t\, \underset{\sim}{u}_t, & t \le T-1 \\ \underset{\sim}{x}'_T\, \underset{\sim}{x}_T, & t = T \end{cases} \tag{2.5}$$

where a is some constant.

The basic problem of control theory is to find a set of control $u_0, u_1, \ldots, u_{T-1}$ so as to optimize J. Solutions to this problem are generally obtained through the technique of dynamic programming. There is an extensive literature on dynamic programming, for a brief description of the technique, the reader is referred to Rustagi (1976).

The dynamic programming technique has been extensively used in various applications. In defense contexts, the technique was suggested by Rustagi and Doub (1970) for optimum distribution of Armor. For an extensive coverage of the art and theory of dynamic programming, see Dreyfus and Law (1977).

In engineering literature, usually a continuous process is considered. Pontriyagin's Maximum Principle has been developed to give a mathematical foundation for such control processes. For a brief description of Maximum Principle and its comparison with Dynamic Programming the reader is referred to Rustagi (1976). Corresponding minimum principles have been advanced by several authors, a recent paper of interest is by Varaiya and Walrand (1980).

Stochastic Control Model

Suppose now the state of a system is given by n-dimensional random vector $\underset{\sim}{x}_t$ at time t. Let $\underset{\sim}{u}_t$ denote the p-dimensional vector of controls at time t and $\underset{\sim}{\varepsilon}_t$ be vector of n-dimensional random disturbances. Usually the control system can be described by the equation

$$\underset{\sim}{x}_{t+1} = g_t(\underset{\sim}{x}_t, \underset{\sim}{u}_t, \underset{\sim}{\varepsilon}_t), \quad t = 1, 2, \ldots, T-1 \tag{2.6}$$

where g_t is a sequence of known functions.

The system with feedback is given in Figure 2.1. In this case the systems deviation from target value already prescribed for the system state $\underset{\sim}{x}_t$ is utilized for adjustment of the system.

In engineering literature usually it is assumed that $\underset{\sim}{x}_t$ cannot be observed directly and instead we observe $\underset{\sim}{y}_t$ with some random error $\underset{\sim}{\eta}_t$. That is,

$$\underset{\sim}{y}_t = h(\underset{\sim}{x}_t, \underset{\sim}{\eta}_t) \tag{2.7}$$

The performance index or objective criterion function of the system now has to be some parameter of the distribution of the cost which is random. In this case, we may optimize $E(J(x_0))$ since $J(X_0)$ is a random variable. The problems of existence and characterizations of optimal controls in the case of stochastic systems, are discussed in detail by Aoki (1967). In the next section, we discuss a linear control process.

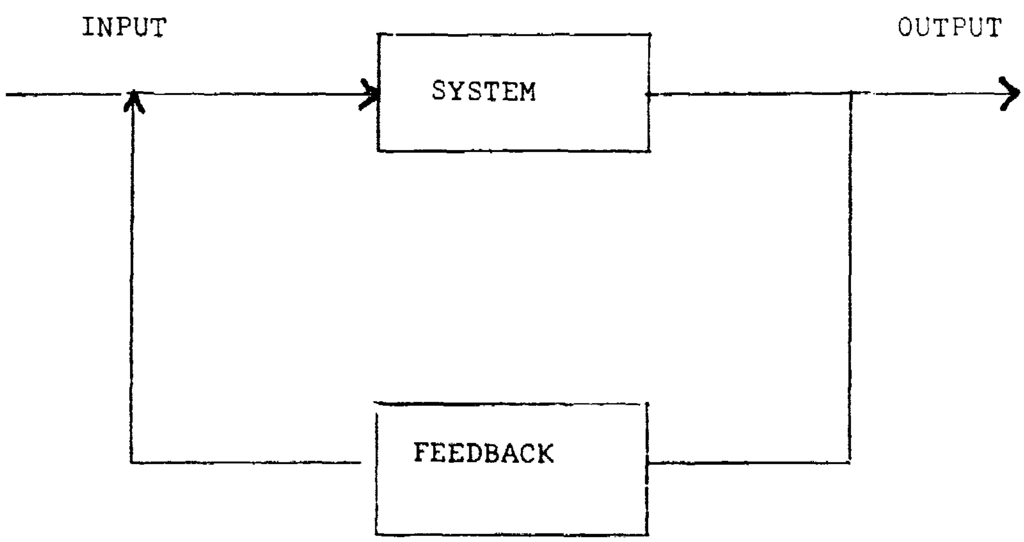

Figure 2.1

3. Linear System and Quadratic Cost

In this section we assume a linear control process and quadratic cost function. We first consider the univariate case. Let

$$x_{t+1} = \alpha_t x_t + \beta_t u_t + \xi_t \quad t = 0, 1, 2, \ldots \tag{3.1}$$

where α_t, β_t are the parameters of the model, x_t is the state of the system at time t and u_t is the control. ξ_t are random errors. Assume that it is not x_t but y_t which is observed. Let

$$y_t = x_t + \eta_t \tag{3.2}$$

where η_t are random errors.

Consider for the example, the cost function to be the terminal control function

$$J = x_T^2 \tag{3.3}$$

Let $\quad \underset{\sim}{y}^{T-1} = (y_0, y_1, \ldots, y_{T-1})$

To optimize E(J), we use in this case, the Principle of Optimality developed by Bellman leading to dynamic programming technique. We optimize $E(x_T^2 | \underset{\sim}{y}^{T-1})$ at first.

We assume here a general form of the error structure and prior information than usually assumed. Let the joint distribution of $(\alpha_t, \beta_t, \xi_t, \eta_t)$ be multivariate normal with mean $\underset{\sim}{\mu}^t$ and covariance $\underset{\sim}{\Sigma}^t$ where

$$\underset{\sim}{\mu}^t = (\mu_\alpha{}^t,\ \xi_\beta{}^t,\ \mu_\xi{}^t,\ \mu_\eta{}^t)'$$

and

$$\underset{\sim}{\Sigma}^t = \begin{pmatrix} \sigma_{\alpha\alpha}{}^t & \sigma_{\alpha\beta}{}^t & \sigma_{\alpha\xi}{}^t & \sigma_{\alpha\eta}{}^t \\ \sigma_{\beta\alpha}{}^t & \sigma_{\beta\beta}{}^t & \sigma_{\beta\xi}{}^t & \sigma_{\beta\eta}{}^t \\ \sigma_{\xi\alpha}{}^t & \sigma_{\xi\beta}{}^t & \sigma_{\xi\xi}{}^t & \sigma_{\xi\eta}{}^t \\ \sigma_{\eta\alpha}{}^t & \sigma_{\eta\beta}{}^t & \sigma_{\eta\xi}{}^t & \sigma_{\eta\eta}{}^t \end{pmatrix}$$

Now

$$\begin{aligned} E(x_T{}^2|\underset{\sim}{y}^{T-1}) &= E(\alpha_{T-1}x_{T-1}+\beta_{T-1}u_{T-1}+\xi_{T-1}|\underset{\sim}{y}^{T-1})^2 \\ &= E[x_{T-1}^2\sigma_{\alpha\alpha}{}^{T-1}+u_{N-1}^2\sigma_{\beta\beta}{}^{T-1}+\sigma_{\xi\xi}^{T-1}+2x_{T-1}u_{T-1}(\sigma_{\alpha\beta}^{T-1}+ \\ &\quad + \mu_\alpha{}^{T-1}\mu_\beta{}^{T-1}) + 2x_{T-1}(\sigma_{\alpha\xi}^{T-1} + \mu_\alpha{}^{T-1}\mu_\xi{}^{T-1}) \\ &\quad + 2u_{T-1}(\sigma_{\beta\xi}^{T-1} + \mu_\beta{}^{T-1} \quad \mu_\xi^{T-1})|\underset{\sim}{y}^{T-1}] \qquad (3.4) \end{aligned}$$

This expression can be further simplified if we are given the conditional distribution of x_t given $\underset{\sim}{y}^{T-1}$. Let

$$E(x_t|\underset{\sim}{y}^t) = \nu_t \qquad (3.5)$$

and

$$V(x_t|\underset{\sim}{y}^t) = \Delta_t{}^2, \qquad (3.6)$$

We then have,

$$\begin{aligned} E(x_N^2|\underset{\sim}{y}^{N-1}) &= \sigma_{\alpha\alpha}^{T-1}(\Delta_{T-1}^2+\nu_{T-1}^2) + u_{T-1}^2\ \sigma_{\beta\beta}^{T-1} + \sigma_{\xi\xi}^{T-1} \\ &\quad + 2u_{T-1}\nu_{T-1}(\sigma_{\alpha\beta}^{T-1}+\mu_\alpha{}^{T-1}\mu_\beta{}^{T-1})+2(\sigma_{\alpha\xi}^{T-1}+\mu_\alpha{}^{T-1}\mu_\xi{}^{T-1})\nu_{T-1} \\ &\quad + 2u_{T-1}(\sigma_{\beta\xi}^{T-1} + \mu_\beta{}^{T-1}\mu_\xi{}^{T-1}). \end{aligned}$$

The optimal control u^*_{T-1} is then obtained by minimizing (3.7) with respect to u_{T-1} given by

$$u^*_{T-1} = -(\sigma_{\beta\beta}^{T-1})^{-1}[\sigma_{\beta\beta}^{T-1} + \mu_\beta^{T-1}\mu_\xi^{T-1} + \nu_{T-1}(\sigma_{\alpha\beta}^{T-1} + \mu_\alpha^{T-1}\mu_\beta^{T-1}) \tag{3.8}$$

Similarly we can follow backwards and obtain u^*_{T-2}.

In the case when α_t, β_t, ξ_t, η_t are assumed to be mutually independent, the covariance $\underset{\sim}{\Sigma}^t$ is diagonal and the expression (3.8) reduces to the one's usually found in text-books, for example, see Aoki (1967) and De Groot (1970).

Bayes Control Policies

In stochastic control theory when the parameters of the model are not known, Bayes methods are commonly used to obtain the optimal control policies. A general formulation of the adaptive Bayes control policies has recently been given by Suzuki (1979).

Consider the following linear model.

$$\underset{\sim}{x}_{t+1} = \underset{\sim}{A}_t \underset{\sim}{x}_t + \underset{\sim}{B}_t \underset{\sim}{u}_t + \underset{\sim}{\xi}_t, \qquad t = 0, 1, 2, \ldots, T-1$$

where $\underset{\sim}{x}_t$ is p-dimensional vector, $\underset{\sim}{u}_t$ is a q-dimensional vector and $\underset{\sim}{\xi}_t$ are random p-dimensional random vectors. Assume further that,

$$\underset{\sim}{y}_t = \underset{\sim}{H}_t \underset{\sim}{x}_t + \underset{\sim}{\eta}_t, \qquad t = 0, 1, 2, \ldots, T.$$

with $\underset{\sim}{y}_t$ being a r-dimensional vector and $\underset{\sim}{\eta}_t$ a random vector. The matrices $\underset{\sim}{A}_t$, $\underset{\sim}{B}_t$ and $\underset{\sim}{H}_t$ are assumed to be known. A general assumption for Bayes analysis made is that the distributions of $\underset{\sim}{\xi}_t$ and $\underset{\sim}{\eta}_t$ are known except some unknown parameters $\underset{\sim}{\theta}_1$ & $\underset{\sim}{\theta}_2$. For simplicity of computations, assumptions are made that $\underset{\sim}{\xi}_t$ have multivariate normal distribution with unknown means $\underset{\sim}{\theta}_1$ but known covariance matrix $\underset{\sim}{\Sigma}_1$, for t = 0, 1, 2, ... T-1. Similarly $\underset{\sim}{\eta}_t$ are assumed to be independent of $\underset{\sim}{\xi}_t$ but distributed normally with unknown mean $\underset{\sim}{\theta}_2$ and known covariance matrix Σ_2. The Bayes strategies are computed by assuming prior distributions on $\underset{\sim}{\theta}_1$, $\underset{\sim}{\theta}_2$ and $\underset{\sim}{x}_0$. In this case, it is assumed that their joint distribution is completely known appropriate multivariate normal distribution. For a the performance function of the quadratic type,

$$J = \sum_{t=1}^{T} (\underset{\sim}{x}_t' \underset{\sim}{P}_t \underset{\sim}{x}_t + \underset{\sim}{u}_{t-1}' \underset{\sim}{Q}_{t-1} \underset{\sim}{u}_{t-1} + \underset{\sim}{y}_t' \underset{\sim}{R}_t \underset{\sim}{y}_t)$$

with known matrices $\underset{\sim}{P}_t$, $\underset{\sim}{Q}_t$ & $\underset{\sim}{R}_t$, the optimal policy can be computed by dynamic programming, first for t = T and then moving backwards. The details are given by Suzuki (1979).

In practical problems suggested earlier, the matrices of the model are not known. However, they can be estimated from data already available by experimenters on those models when the process under control is observed for a sufficiently long period of time. This aspect of the control problem is discussed in the next section.

5. Statistical Considerations

Consider the case when the control process has been observed for a certain period of time. Assuming the form of the dynamics of the process, such as linearity, the unknown parameters of the process equations can now be estimated from the data. These estimates can then be used for statistical control.

Suppose the model at time t is

$$x_{t+1} = \alpha_t x_t + \beta_t u_t + \varepsilon_t \tag{5.1}$$

with assumptions on the errors ε_t. For illustration, consider the situation when ε_t's are independently and identically distributed with means 0 and variance σ^2, and for estimation purposes, the parameters α_t and β_t have remained constant during t = -T, - T+1, ..., -1, 0, say, equal to α_0 and β_0 respectively. In this case the least squares estimates are given by the following normal equations, where the summation is from t = -T to t = 0.

$$\begin{pmatrix} \sum x_t^2 & \sum x_t u_t \\ \sum x_t u_t & \sum u_t \end{pmatrix} \begin{pmatrix} \hat{\alpha}_0 \\ \hat{\beta}_0 \end{pmatrix} = \begin{pmatrix} \sum x_{t+1} x_t \\ \sum x_{t+1} u_t \end{pmatrix} \tag{5.2}$$

It is well known that under the additional assumptions of normality of errors ε_t, the above estimates are also maximum likelihood estimates of α_0 and β_0. The general system of equations when the model contains more parameters can be similarly obtained, for example see Anderson (1971, p. 183).

In case $\underset{\sim}{x}_t$ is a p-dimensional vector, and $\underset{\sim}{u}_t$ is a q-dimensional vector of controls, we consider the model

$$x_{t+1} = \underset{\sim}{A}_t \, x_t + \underset{\sim}{B}_t \, u_t + \underset{\sim}{\varepsilon}_t, \tag{5.3}$$

Here $\underset{\sim}{A}_t$ is a p x p matrix of unknown parameters, $\underset{\sim}{B}_t$ is a p x q matrix of unknown parameters and $\underset{\sim}{\varepsilon}_t$ is a p-dimensional vector with mean 0 and covariance matrix $\underset{\sim}{\Lambda}$. The normal equations in such a case are given by

$$\begin{pmatrix} \sum \underset{\sim}{x}_{t-1} \underset{\sim}{x}_{t-1}' & \sum \underset{\sim}{x}_{t-1} \underset{\sim}{u}_t \\ \sum \underset{\sim}{u}_t \underset{\sim}{x}_{t-1}' & \sum \underset{\sim}{u}_t \underset{\sim}{u}_t' \end{pmatrix} \begin{pmatrix} \hat{\underset{\sim}{A}}_0' \\ \hat{\underset{\sim}{B}}_0' \end{pmatrix} = \begin{pmatrix} \sum \underset{\sim}{x}_{t-1} \underset{\sim}{x}_t' \\ \sum \underset{\sim}{u}_t \underset{\sim}{x}_t' \end{pmatrix} \tag{5.4}$$

under the same assumptions as in the univariate case and the estimates of $\underset{\sim}{\Lambda}$ are given by

$$\Lambda = \frac{1}{T} \sum (\underset{\sim}{x}_t - \hat{\underset{\sim}{A}} \, \underset{\sim}{x}_{t-1} - \hat{\underset{\sim}{B}} \, \underset{\sim}{u}_t)(\underset{\sim}{x}_t - \hat{\underset{\sim}{A}} \, \underset{\sim}{x}_{t-1} - \hat{\underset{\sim}{B}} \underset{\sim}{u}_t)'$$

The detailed development of the estimates and the associated theory is given by Anderson (1971, p. 203).

The control process will utilize the estimates of $\hat{\alpha}_0$ and $\hat{\beta}_0$ as the initial estimates and use them to update the estimates as the data accumulates at each stage of the decision making process as t goes from 0 to T. Some simpler forms of recursive estimates, given by Albert and Gardner (1966) and Albert (1972) are discussed next. They can be implemented in any adaptive estimation scheme.

Recursive Estimates

Most of the work in control theory in engineering applications is confined to the case when the parameters of the model are known and are not modified during the operation of the control process. When continuous modification of the joint density of the unknown parameters can be made as new observations become available, the calculations become difficult. In place of using the Bayesian approach that prior density of the parameters is given, we use henceforth the classical estimate the parameters as given in (5.2) or (5.4). We use the recursive estimation procedure for the parameters using observations from the process and the control during the operation of the process.

Let the linear model considered in equation (5.1) be simplified by using $\underset{\sim}{\theta}_t = (\alpha_t, \beta_t)$ and $\underset{\sim}{h}_t = (x_t, u_t)$ so that (5.1) can be written as

$$x_{t+1} = \underset{\sim}{h}_t' \underset{\sim}{\theta}_t + \varepsilon_t \tag{5.5}$$

Let $\hat{\underset{\sim}{\theta}}_t$ is the estimate of $\underset{\sim}{\theta}_t$ based on (t-1) observations. x_t is the observed state at time t. We would like a procedure to updata $\hat{\underset{\sim}{\theta}}_t$ based on x_t.

<u>Definition</u>: $\hat{\underset{\sim}{\theta}}_{j+1}$ is a <u>recursive</u> estimate of $\underset{\sim}{\theta}$ if $\hat{\underset{\sim}{\theta}}_{j+1}$ is obtained by updating $\hat{\underset{\sim}{\theta}}_j$ by the observation x_j. That is,

$$\hat{\underset{\sim}{\theta}}_{j+1} = g_j(\hat{\underset{\sim}{\theta}}_j, x_j) \tag{5.6}$$

The suggested recursive estimate of the differential type by Albert and Gardner (1967, p. 111) is the following.

$$\hat{\underset{\sim}{\theta}}_{j+1} = \hat{\underset{\sim}{\theta}}_j + \underset{\sim}{a}_j(x_j - \underset{\sim}{h}_j'\hat{\underset{\sim}{\theta}}_j) \tag{5.6}$$

with

$$\underset{\sim}{a}_j = \underset{\sim}{B}_j \underset{\sim}{h}_j \tag{5.7}$$

and

$$\underset{\sim}{B}_j = \underset{\sim}{B}_{j-1} - \frac{(\underset{\sim}{B}_{j-1} \underset{\sim}{h}_j)(\underset{\sim}{B}_{j-1} \underset{\sim}{h}_j)'}{1 + \underset{\sim}{h}_j' \underset{\sim}{B}_{j-1} \underset{\sim}{h}_j} . \tag{5.8}$$

The above recursive estimates can be written in closed form if we are given the initial estimates $\hat{\underset{\sim}{\theta}}_0$ and $\underset{\sim}{B}_0$. In that case

$$\underset{\sim}{B}_k = (\underset{\sim}{B}_0^{-1} + \sum_{j=1}^{k} \underset{\sim}{h}_j \underset{\sim}{h}_j')^{-1}, \quad k = 1, 2, \ldots \tag{5.9}$$

and

$$\hat{\underset{\sim}{\theta}}_{j+1} = \underset{\sim}{B}_j(\underset{\sim}{B}_0^{-1} \hat{\underset{\sim}{\theta}}_1 + \sum_{i=1}^{j} h_i x_i), \quad j \geq 0 \tag{5.10}$$

The convergence and other optimal properties of these recursive estimates are enumerated by Albert and Gardner in their book.

A recent survey of recursive estimation procedures has been given by Davis (1977) utilizing concepts of innovative sequences due to Kailath (1968). However, most of this survey is concerned with continuous processes.

Assuming now that the process equations are being updated according to the procedure described above, in equations (5-6) - (5.10), we assume that the model for control is now the following

$$x_{t+1} = \hat{\alpha}_t \, x_t + \hat{\beta}_t \, u_t \qquad (5.11)$$

Suppose

$$\underset{\sim}{x}^{t-1} = (x_0, x_1, \ldots, x_{t-1})$$

$$\underset{\sim}{u}^{t-1} = (u_0, u_1, \ldots, u_{t-1}),$$

and t = 1, 2, ..., T.

Let the performance function for this process be the following

$$J = \sum_{t=0}^{T-1} (a_t \, x_t^{\,2} + b_t \, u_t)^2 + c. \, x_T^2 \qquad (5.12)$$

where a_t, b_t and c are known for t = 0, 1, ..., T-1. The optimization procedure is concerned with finding u_0, u_1, ..., u_{T-1} so as to minimize

$$E(J|\underset{\sim}{x}^{T-1}, \underset{\sim}{u}^{T-1})$$

Let $V_t(x_t)$ = minimum cost if the process is in state t with state variable x_t. Note that $V_t(x_t)$ and the discussion what follows is conditional on $\underset{\sim}{x}^t$ and $\underset{\sim}{u}^t$. This fact is not expressed in the notation.

Using Principle of Optimality of dynamic programming,

$$V_t(x_t) = \min_{u_t}\{ a_t \, x_t^{\,2} + b_t \, u_t^{\,2} + V_{t+1}(\hat{\alpha}_t \, x_t + \hat{\beta}_t \, u_t)\}$$

t = 1, 2, ..., T-1, and

$$V_T(x_T) = c \, x_T. \qquad (5.13)$$

Starting backwards, we first consider $V_{T-1}(x_{T-1})$.

$$V_{T-1}(x_{T-1}) = \min_{u_{T-1}} \{a_{T-1}\, x_{T-1}^2 + b_{T-1}\, u_{T-1}^2$$

$$+ c(\hat{\alpha}_{T-1}\, x_{T-1} + \hat{\beta}_{T-1}\, u_{T-1})^2\}$$

Hence the optimal value of u_{T-1} is obtained as follows.

$$u_{T-1}^* = \frac{c\hat{\alpha}_{T-1}\, \hat{\beta}_{T-1}\, x_{T-1}}{b_{T-1} + c\hat{\beta}_{T-1}^2} \tag{5.14}$$

and the corresponding value of the cost function is given by

$$V_{T-1}^*(x_{T-1}) = (a_{T-1} + c\hat{\alpha}_{T-1} - \frac{c^2\hat{\alpha}_{T-1}\, \hat{\beta}_{T-1}^2}{b_{T-1}+c\, \hat{\beta}_{T-1}^2})x_{T-1}^2 \tag{5.15}$$

or

$$V_{T-1}^*(x_{T-1}) = \hat{p}_{T-1}x_{T-1}^2 \tag{5.16}$$

$\hat{p}_{T-1}$ is the coefficient of x_{T-1}^2 in (5.15). Similarly, we obtain

$$u_{T-2}^*\ (x_{T-2}) = \frac{\hat{p}_{T-1}\ \hat{\alpha}_{T-2}\ \hat{\beta}_{T-2}\ x_{T-2}}{b_{T-2} + \hat{p}_{T-2}\ \hat{\beta}_{T-2}^2}\ ,$$

and

$$V_{T-2}^*(x_{T-2}) = \hat{p}_{T-2}\ x_{T-2}^2. \tag{5.17}$$

In general, one obtains the optimal controls following the above process given by

$$u_t^*(x_t) = - \frac{\hat{p}_{t+1}\ \hat{\alpha}_t\ \hat{\beta}_t\ x_t}{b_t + \hat{p}_{t+1}\ \hat{\beta}_t^{\,2}} , \tag{5.18}$$

$$V_t^*(x_t) = \hat{p}_t\ x_t^{\,2}, \tag{5.19}$$

with

$$\hat{p} = a_t + \hat{p}_{t+1}\ \hat{\alpha}_t^{\,2} - \frac{\hat{p}_{t+1}^2\ \hat{\alpha}_t^{\,2}\ \hat{\beta}_t^{\,2}}{b_t + \hat{p}_{t+1}\ \hat{\beta}_t^2} . \tag{5.20}$$

$$t = 1, 2, \ldots, T\text{-}1 \qquad (5.20)$$

and

$$\hat{p}_T = c. \tag{5.21}$$

Many other performance functions can be similarly used to obtain the optimal policy. Large number of such problems in the deterministic case are discussed in a recent survey on art and theory of dynamic programming by Dreyfus and Law (1977).

The stochastic behavior of the optimal control policy seems fairly complicated in general. In case of U_{T-1}^* given by (5.13), we find that it is approximately the ratio of the product and square of random variables which themselves have a complicated distribution. In practical situations, we shall assume that the number of observations on which the estimates of α_t and β_t are based, is large. Therefore, using asymptotic theory, we have that $\hat{\alpha}_t \to \alpha_t$ and $\hat{\beta}_t \to \beta_t$ in probability, allowing us to use the optimal control policy.

These estimates or their generalizations are applicable to various applications discussed earlier.

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